

Wittgenstein's Construal of "Numbers" as "Schemes" and the Color Incompatibility Problem

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INTRODUCTORY REMARKS

THE PROBLEM'S SETTING

This paper deals with a problem faced by Wittgenstein in the *Tractatus Logico-Philosophicus*, in his unpublished *Manuscripts* 105 and 106, and in his only published paper – “Some Remarks on Logical Forms” – (SRLF, for now on).¹ The problem was named by Wittgenstein's interpreters as the “color incompatibility problem”, for it has to do with the reduction of predicates of color to basic units of brightness, intensity, and Chroma.

The color incompatibility problem consists of an unexpected impediment faced by Wittgenstein with the reduction of colors' predicates to a list of more basic ingredients, which should number each color's components. That difficulty appeared during Wittgenstein's efforts to establish the logical possibility of what we have called in another paper (VELLOSO, 2014) his “Great Analysis” (GA, from now on). The procedure, a characteristic of Wittgenstein in the *Tractatus*, consists in analyzing all ordinary propositions, translating them into a disjunction of conjunctions of Elementary Propositions (EPs, from now on), which would lie at the bottom of the linguistic hierarchy. At GA's final phase, all propositions should have had their meaning completely determined. GA's goal is also known in the secondary literature as the “principle of a complete determination of sense” (PCDS, from now on). The color incompatibility problem was such an insurmountable obstacle that it could block all the available routes to establish the possibility, even in principle, of reaching GA's final phase and consequently the complete determination of sense also.

Wittgenstein's enunciation of the PCDS was very clear and direct. As he put it in the *Tractatus*, the PCDS is the consequence of connecting two ideas: “the possibility of simple signs” and “the determination of sense”: “The requirement that simple signs be possible is the requirement that sense be determinate.” (1961, 3.23) In another aphorism relevant to understanding PCDS, the aphorism 4.022, Wittgenstein affirmed that “A proposition shows its sense.

1 The manuscripts 105-106, were posterior to the *Tractatus*, dated from the first half of the year 1929, and contemporaries to the SRLF. Those manuscripts are transcribed in full to the (WITTGENSTEIN, Wiener Ausgabe Studien Texte (Band 1 - 5) 2001). The manuscript 105 is in Band 1, and 106 in Band 2. We are going to quote the manuscripts' passages translated by us from the *Wieber Ausgabe Studien Texte*.

A proposition shows how things stand if it is true. And it says that they do so stand". And again, in *Manuscript* 106 we find:

3. / If a sentence is to have a specific meaning (and otherwise it would be nonsensical) its sense must be fully graspable – completely visualizable; the generality only makes sense if it - i.e., all values of the variables - is completely determined. [my translation] (2001, 85, Band 2)

It follows from those passages that Wittgenstein's goal of a complete determination of all propositional senses must be identified with the complete determination of all propositions' truth-conditions, including the general ones. So, besides blocking the completion of the GA, and making the PCDS unsustainable, the color incompatibility problem also dismantles the idea of a complete determination of all propositions' truth-conditions.

In the *Tractatus*, Wittgenstein had said that this first goal of GA, the complete determination of sense, was dependent on establishing the possibility of a 1-1 logically independent connection of each EP to its unique atomic state-of-affairs (Atomic SoAs, from now on). In the final phase of the GA, all EPs should have their truth-conditions completely determined in an independent way.

In SRLF, the only paper written by Wittgenstein, he presents a kind of summary of his investigation regarding the color incompatibility problem, developed in detail in the intermediary period of his work from 1929 until 1933 in the *Manuscripts* 105 and 106. In the former paper, Wittgenstein presents the results of his discussions in the *Manuscripts*. The philosopher admits there that his goal was untenable, contrary to what he has thought before. The difficulty he encountered with the elimination of colors' predicates was so insurmountable that at the end of the paper Wittgenstein finally decided to abandon his first and more radical claim about EPs – the requisite of their logical independence – and go ahead without it, trying to still explain EPs' sense, but now in terms of a primary, purely phenomenological language.

Wittgenstein's confidence in the possibility of having a foundational language (the one he later calls a "primary language") that would truth-functionally completely determine the sense of every proposition from the base to the most general top-level was a very firm philosophical credo, despite the difficulties he found with the reduction of color predicates. In the *Manuscripts*, where Wittgenstein was clearer about the impasse he has gotten into, he evaluates another alternative not discussed in SRLF to achieve his goal of a primary foundational language.

Having given up the more radical goal of a complete determination of sense for fully independent EPs, Wittgenstein had still tried to carry out a similar goal, this time based on a weaker goal: to find a primary language that copes with the PCDS in its more restrained formulation. This time, the determination of sense for the top of the hierarchy would still be done truth-functionally from the singular propositions, but there would no longer be the genuinely elementary propositions. Colors and other similar predicates would be reduced to coordinate numbers, but the procedure would involve dealing only with internal relations which held the constituents' elements within each EP together. This last change, however, implied that he would no longer see the obtainment of numerical coordinates for colors truth-functionally. Or in other words, he would be relinquishing the possibility of reaching the final phase of the GA with genuine EPs. So, in the internal context of EPs, one would find only schemes. Those figurative structures would have the sole task of displaying numerical values of phenomenological experiences.

OUR STRATEGY

Against this background, our task will be a twofold one. First, we aspire to show how and why Wittgenstein got stuck into that dilemma. This will be the paper's first goal. We believe that after having explained in detail Wittgenstein's motivations and his strategy for validating PCDS, we will have a better view of the entire investigation carried on in the manuscripts. As a preliminary step for that first task, we think it is crucial to show how Wittgenstein's full-hearted adoption of Frege's notion of "sense as truth condition" was at the root of all those difficulties.

Our second task will be much more specific. We will try to elucidate the impossibility of reducing color predicates to more fundamental units of brightness, chroma, and intensity. The aim is to show how and why numbers for colors had to be introduced into the EPs. We also want to show how this could be done only through the idea of a "scheme", according to Wittgenstein. This structure would have the purpose of explaining numbers as the result of an operation that would not be truth-functional, for this operation would happen inside the inner formal structure of each isolated elementary proposition. We also intend to present some advantages Wittgenstein obtained from using a special kind of notation for his construal of these "schemes", the *bar notation*.

For achieving our second goal, our strategy will be to compare Wittgenstein's solutions involving the *bar notation*, discussed in the *Manuscripts*, with Frege's systematic discussion and critique of this same topic in the *Grundlagen*. With the background of Frege's investigation, we hope to clarify Wittgenstein's objectives with his alternative proposal of seeing "numbers" as "schemes", offered as a second solution for the colors' problem.

I. THE NOTION OF "SENSE" AS "TRUTH-CONDITIONS" AND ITS THEORETICAL COST

I.1 A COMPARISON BETWEEN FREGE'S AND WITTGENSTEIN'S CONSTRUALS OF "SENSE" AS "TRUTH-CONDITIONS"

The assignment of a fregian origin to the tractarian notion of "sense as truth-conditions" is not above reasonable doubt. The assignment of its authorship to Frege was eventually disputed among some commentators.² However, it is not an uncommon thesis either. Michael Dummett, for example, in his book *The Seas of Language*, says that Frege has explicitly contended for the notion of "sense" as "truth-conditions".

Does the meaning of a sentence consist in its truth-condition? Does the meaning of a word consist in the contribution it makes to determining the truth-condition of any sentence in which it occurs? [...] it has been explicitly contended for by Frege, by the Wittgenstein of the *Tractatus*, and by Davidson. (1996, 33-34)

We will side with Dummett and just assume it. In our opinion, there are other important aspects present in both Frege's and Wittgenstein's approaches which are more relevant to our discussion here.

The first aspect which was shared by Frege's and Wittgenstein's proposals was that both have adopted in their own way a kind of analytical method for showing how the idea of "truth-condition" should be treated at the structural level of singular propositions. They also

2 For a dispute on this matter, see (TREBAUL Sep 2012).

shared the same goal: to reach a basic level where sense is completely determined. The main difference between them, though, is that Frege's project involved the setting up of a complete extensional and set-theoretical semantics, while Wittgenstein's did not.³

In *Grundgesetze*, Frege implemented his own foundational semantics by stipulating in the first ten paragraphs of the book three distinct kinds of functions that should work together in the accomplishment of establishing a universal domain of logical objects: the "value ranges operator", "the backslash operator" and the "smooth-breathing operator". To simplify the presentation of Frege's ideas, let us group those functions altogether into a cluster and call them "the extensionality operations". The main task of those "extensionality operations" was the introduction of one representative at level zero for each property of any higher order degree. According to Frege, those extensions should then supply a clear-cut *identity* criterion for all concepts. For him, the extensionality operations should be sufficient to establish the truth value of whole propositions as a function of the sense of their component's expressions.

Frege's analytical method did not have to be a process as long as that proposed by Wittgenstein, however. His plan was just to get a single universal domain of logical objects and to use it as semantical support for the entire system. In the *Tractatus*, on the other hand, Wittgenstein wanted to go further with his analytical method.⁴ His goal was to reach the lowest and simplest part into which all possible events could resolve themselves. Or, put in other words, Wittgenstein did not find the predicative form to be appropriate for EPs, nor has he accepted Frege's universal domain of logical objects at level zero. Instead, he chose to implement a longer process of analysis, the GA, to uncover the hidden logical form of all EPs. With this accomplishment, the new conceptual notation would be completely free of indetermination, dependent only on the construal of "propositional sense" as "truth-conditions".

In our opinion, the most important trace distinguishing Wittgenstein's and Frege's plans is precisely the universality or non-universality of the predicative logical form, adopted by the latter and refused by the former. Due to this reason, the elder philosopher, but not the youngest, was committed to the idea of analyzing the propositional structure just to the point where we find the extensions as the sole representant of linguistic expressions in the universal domain of his semantics.

I.2 FREGE'S UNIFICATION OF TRUTH AND GRAMMAR IN ONE SINGLE UNIVERSAL LANGUAGE

It is the goal of this section to show how Frege had intended to unify truth and grammar into one single universal and predicative logical form, which should be applied to the entire language. To show that, we will begin with Frege's explanation of logic's task, embedded in this famous passage coming from his later works:

The word "true" indicates the aim of logic as does "beautiful" that of aesthetics or "good" that of ethics. All sciences have truth as their goal, but logic is also concerned with it in a quite different way from this. It has much the same relation to truth as physics has to weight or heat. To discover truths is the task of all sciences; it falls to logic to discern the laws of truth. (FREGE, *The Thought: a logical inquiry* 1956, 289)

3 We are not intending to give details of both methods, for they will lead us far from the purpose of the paper. Our intention was only to emphasize the differences and to connect these differences with Wittgenstein's alternatives.

4 In another forthcoming paper we argued for the thesis that Wittgenstein's aim of going further with his analytical process was due to the failure of Frege's notion of "extension" and the paradox it involves.

In this passage, Frege adopted the view that logic is concerned mainly with “discerning the laws of Truth”. For him, “sense” and “truth” are so intimately connected that there could be no question of separating them into two parts, one concerned with semantics and meaning, and the other with the laws of truth themselves. Despite that, as we will see, that separation could still be implemented with fruitful results.

In another paper, we’ve suggested that the discipline of logic could be separated into two distinct levels: the propositional level and the predicate level. We’ve argued there that those two logical levels could be treated separately following two different approaches.⁵ The *Propositional logic* would be concerned with laws of truth and it would be left to *Predicate logic* mainly to supply a metaphysical interpretation to objects and properties.⁶

As we’ve said, the division was not relevant for Frege. He saw logic as a single unit. In his view, the logic of propositions would be just a secondary consequence of his predicative analysis. As a result, his analytical process aimed to just define exact criteria for sense. Frege’s construal of “propositional sense” however was directly connected with the “discernment of the laws of truth”. For him, predicate logic should be the most basic logic and should be intimately concerned with “sense” and “truth”. To propositional logic, it was left the task of dealing with the laws of truth themselves. He explained how truth should be considered as a consequence of “propositional sense” by evoking the idea of “satisfaction of truth-conditions”. So, the two approaches to logic were essentially connected in his universal logical system by the idea of “a universal logical grammar”.

Going counter to Frege’s proposals, in the *Tractatus* Wittgenstein suggested another way of understanding language. His first move was very peculiar: he attributed the aprioristic character of logic exclusive to the propositional explanation of the logical space in terms of truth tables. Thereafter, he delegated the task of supplying a metaphysical explanation of reality, not to predicate logic, as would be normally expected, with its variables, quantifiers, predicates, and sets, but to the sub-propositional and internal structure of EPs. As we have claimed in that other paper⁷, it is at least not transparent to the reader why the sub-propositional analysis of singular EPs in the *Tractatus* should involve something even close to a set-theoretical approach. Quite on the contrary, we have tried to show in that other paper⁸ that, at the lowest level of language, the *Tractatus* explanation of EPs’ internal structure involved a mereological kind of explanation.

Wittgenstein’s GA method of getting to the sub-propositional level of EPs also differs in essential features from those other strategies mentioned here, like Frege’s, for example. The primary point of divergence is that in the *Tractatus*, Wittgenstein did not make those same assumptions about the logical form of the most singular propositions of language. At this foundational level, we find another sort of analysis being implemented, diverse from the predicative form. Instead of two parts, the logical subject and the logical predicate, as in Frege, he proposed another way to view the EP’s internal structure and another way to analyze the internal relation between their elements.⁹ So, in the *Tractatus*’s ground level, we find no “sets”, as we do in Frege’s universal domain.¹⁰

5 Velloso, “Frege and Wittgenstein’s debate regarding de notion of “fact” in the *Tractatus*. *Is it a Set-theorist or a Mereological notion?*”, forthcoming.

6 Ibidem.

7 Velloso (“Frege and Wittgenstein’s debate regarding the notion of “fact” in the *Tractatus*”, forthcoming)

8 Ibidem.

9 Following Chateaubriand, we will call “what it is said about” as the logical-subject, and “what is said” as the logical-predicate. As he also points out, this distinction became clearer after Frege’s paper “On sense and reference”. (CHATEAUBRIAND 2001, 240-244, cap 2)

10 In Velloso, “Frege and Wittgenstein’s debate regarding the notion of “fact” in the *Tractatus*”, forthco-

I.3 THE PRINCIPLE OF THE LOGICAL INDEPENDENCE OF ELEMENTARY PROPOSITIONS

According to Wittgenstein, there was a principle which should be valid for the zero level of EPs but was still poorly understood regarding its importance. We are talking about the Principle of the Logical Independence of Elementary Propositions (PLIEP, for now on) formulated for elementary propositions and atomic facts. In aphorism 2.061 – 2.062, Wittgenstein exposed the principle, considering the case of atomic facts.

2.061 Atomic facts are independent of one another.

2.062 From the existence or non-existence of one atomic fact it is impossible to infer the existence or non-existence of another. (1961)

The PLIEP presupposes an independent connection between each elementary proposition and its figured atomic SoA. Or, put it in other words, the PLIEP presupposes the completeness of Wittgenstein's GA. This means that there must be one single way to reduce all propositional content, through translational procedures, to the layer of genuine EPs. This is a precondition for achieving the PLIEP. The PLIEP was so fundamental for Wittgenstein that, without it, the whole idea of "sense" as "truth condition" would be lost, as we have said in the introduction.

Wittgenstein's GA began as a simple idea, though, a mere translational process. In fact, all logical positivists have had that same idea, the removal of all the ambiguous and problematic predicates from our ordinary language and their replacement by logical expressions endowed with sharp conditions of applicability. With this "repairing method", they contemplated the obtainment of a purely empirical language, composed only of sharply determined concepts. The regimentation Wittgenstein had in mind, however, involved much more than a simple translational and cleaning process. It was the hardest and more radical reductionist process ever tried. It aimed to establish a unique connection of all propositional contents to an atomic and foundational ground.

The implementation of that process should supply us with the totality of all elementary propositions. Only then it would be possible to recover all linguistic generality added by truth-functional operations upon that elementary basis. As Wittgenstein puts it in the following passage:

4.52 Propositions comprise all that follows from the totality of all elementary propositions (and, of course, from its being the totality of them all). (Thus, in a certain sense, it could be said that all propositions were generalizations of elementary propositions.)

5.3 All propositions are results of truth-operations on elementary propositions. (1961)

Whatever corresponds in reality to compound propositions must not be more than what corresponds to their several atomic propositions. Molecular propositions contain nothing beyond what is in their atoms; they add no material information above that contained in their atoms. [...] In fact, the understanding of general propositions obviously depends on that of atomic propositions. (1957, 238)

We believe that in those passages Wittgenstein was not just asserting that all generality comes from a singular basis and were true or false depending on it. He was also embracing

ming, we concluded that the relation between simple objects and Atomic SoAs is that of a part and its mereological aggregate, and not that of an element and a set.

the idea that propositions other than EPs did not say anything more than a collection of EPs put together. Based on those conclusions, we can look at the PLIEP as a way of linking the two approaches to logic we have proposed, the propositional level of truth functions and the level of logical analysis that would be directly concerned with the EPs. For Wittgenstein, it is this last logical level that should be entrusted with the metaphysical task of explaining sense and truth-conditions.

Unfortunately, those results depended on keeping the PLIEP. Indeed, the very success of the GA would depend on this principle. In the next section, we will try to understand how EP's logical form could fit within that radical reductionist scenario.

II. THE SUB-PROPOSITIONAL LEVEL IN THE TRACTATUS

II.1 EPs AS CONCATENATIONS OF NAMES

One of Wittgenstein's main difficulties was precisely to give examples of EPs in the *Tractatus* (1961, 3.201, 3.23). Despite the absence of examples, he underscored many times in the *Tractatus* the importance of arriving at the right analysis of these fundamental linguistic entities. His concerns there were that, to guarantee the universal validity of the PCDS, he needed a complete understanding of this ultimate foundational layer.

In the *Tractatus*, Wittgenstein gave us a small glimpse of how to conceive EP's logical form. They should look like a kind of chain or sequence of tractarian simple names concatenated by an internal relation.

4.22 The elementary sentence consists of names. It is a connection, a concatenation (*verkettung*), of names. (1961)

The first consequence of Wittgenstein's assertions in this aphorism is that we should get rid of all general terms from inside the EPs. As we discussed in that other paper, Wittgenstein thought those expressions were impregnated with semantical generality.¹¹ The goal he set himself, however, was not an easy one. He would have to translate all predicative expressions which express semantical generality of any kind whatsoever into sequences of simple signs. A first concern about the enormous translational work involved in this procedure is its result: at GA's final phase, EPs would become a simple concatenation of *tractarian* "names".

One question at once impose itself: how the bare concatenation of those simple signs would manage to present an atomic SoA in the logical space? In our opinion, any outline of an answer to this question should begin with a comment made by Wittgenstein in the *Tractatus*.

2.15 The fact that the elements of a picture are related to one another in a determinate way represents that things are related to one another in the same way. Let us call this connection of its elements the structure of the picture and let us call the possibility of this structure the pictorial form of the picture. (1961)

According to Wittgenstein in this passage, the relation between language and reality should be based on the logical possibility of correlating those two isomorphic structures: one linguistic and the other ontological, a possibility in the logical space.¹² At the point where

11 Cf. footnote 10.

12 Usually, an isomorphism is understood as a relation between two languages, where the truth of one can be transferred, *salva veritate*, to the other. But one could also understand "isomorphism" as it is frequently

language and ontology meet, one would be left only with the singular relationship connecting each EP with its correspondent atomic SoA. Still, according to Wittgenstein, this relationship also depends on both sides having the same "mathematical multiplicity". What he probably meant by this expression was that, in both cases, the linguistic and the ontological side should have the same number of elements arranged in the same structural way.

Another issue worth commenting on here is the unusual idea of "analyzing away" all general terms from EPs.¹³ After all, why should one employ this problematic explanation concerning the logical form of the most singular propositions of the system? Why should one get oneself tangled with many problems?

We want to elaborate on a first attempt to answer those questions. As we've said already, Wittgenstein's difficulties with predicates had to do with a kind of "semantical generality" he thought they had. We are thinking here about the fact that the semantical task of general terms is not to name (or simply point to) one single atomic "object". In fact, more than one complex of objects could offer itself as the one presented by that expression. So, whenever one of those expressions was present, the analytical process of translating it to a more detailed description of the situation that would make the whole sentence true has to move on further. The goal is to reach the point where each EP has its own singular true maker. Another way to put the same point is to say that, as long as there is more than one situation that could make some proposition true, we have not yet reached the final phase of the process. The desired result would be to find the genuine EPs, those that indeed have true makers of their own.

This problem of a proposition that could have been made true by more than one situation is just one side of the difficulty. The other side is that once the subject matter of the statement is fixed and it does not involve a general term, but only a truly singular one, we must turn our attention to the assertion that is being made. At this point, we have several alternatives. The first one is offered by the universal semantics of set theory. This was not what Wittgenstein had in mind, though. He was looking for an explanation that takes care of any case of indetermination caused by some general aspect in the situation described.

The scenario Wittgenstein had wished for is the one of a complete atomization of all expressions. He was thinking of any singular situation in the sense of it being completely unrepeatable. On this account, he thought it must be possible to list each atomic component of an event, together with their connections. These "analyzed away" general terms should be retrieved later on in a methodical way, by a logical operation of conjugation applied to EPs. This means one should fix her attention on those atoms and the internal relations that hold them together.¹⁴ By attentive observation, one should be led to conclude that the atomic description presented by a logical product of EPs is a description of such-and-such event (as they are normally characterized in the ordinary terms of our natural languages).

Summarizing our argument in this section, at the final phase of the GA all ordinary predicates must have been replaced by the right sequence of individual ingredients. At this point, any theoretical formulation of predicates as classes would be illegitimate. In such an extraordinarily reductionistic scenario, the PLIEP would be simply an immediate consequence of displaying the simpler ingredients from which all complexes are composed.

used in the *Tractatus'* secondary literature, as a kind of correlation between language and reality. So, we understand "isomorphism" as the possibility of establishing a function that correlates the two structures, "the structure of the picture" and "the way things are related to one another" in the logical space.

13 Cf. (VELLOSO, Wittgenstein's unique "Great Analysis": a consequence of the construal of propositional sense as truth-conditions 2014).

14 We are going to deal with the construal of "internal/external relations" in next section.

One last remark. Wittgenstein's EPs and the Atomic SoAs figured by them should be localized as they were below the logical level occupied by Frege's singular propositions and by the universal domain of extensions, respectively. They should be the end of our search for a complete determination of sense, the ultimate ground where all the other linguistic levels were grounded. Next, we will discuss some commentators of the *Tractatus* who adopted this same viewpoint about EPs' internal structural form.

II.2 EPs AS SEQUENCES OF NAMES IN THE LITERATURE

James Griffin had already singled out the tractarian GA's goal of analyzing away all predicative expressions from EPs. In his book *Wittgenstein's Logical Atomism*, he says that all descriptive properties should disappear on analysis, and, in their turn, only *simple* names should appear in the internal configuration of EPs. At the final phase of GA, one will get only sequences of interconnected simple names.

Wittgenstein's thesis is a universal one: all properties disappear on analysis. Perhaps Wittgenstein intends his remark on the incompatibility of colors to apply equally widely. That is, it may be part of his argument that for any property, there is at least one other to which it stands in internal relation, which pair of properties logically exclude one another. (GRIFFIN 1964, 79)

Griffin's opinion is that any kind of grammatical generality, like the one all qualitative properties have, would again bring about the same devastating consequence color words brought in the first place: "pair of properties [which] logically excludes one another". Consequently, their very presence causes the loss of the necessary logical independence of EPs. This loss at once puts at risk Wittgenstein's particular construal of "propositional sense" as "truth conditions". Later on, in a passage from *The Big Typescript* (BT, from now on), Wittgenstein acknowledges that this point went unnoticed by him in the *Tractatus*:

[...] but what I overlooked was that if this transformation of the proposition $f(a)$ were to consist in its being replaced by a logical product, then the factors of this product would have to have an *independent sense* [...]. (2005, 340e)

At the *Tractatus*, but not in the *Manuscripts* 105 and 106, or in SRLF, the logical product of elementary propositions must be composed of ingredients with an independent naming relationship with their ontological correlates. Therefore, the names of those ingredients could not be general terms, which have to be inferentially related to the meaning of other expressions as well.

Another author who embraces this interpretation is Guido Küng in the Book (Ontology and the Logistic Analysis of Language 1967). In chapter 6 of part 2, dedicated to the *Tractatus*, he emphasizes the idea that "Wittgenstein might have explained the predicate signs of non-ideal languages" through the "Actual analysis [which] reveals [the actual] logical forms" (1967, 83) of EPs. Furthermore, in a section called "The ideal language without predicate signs", he commits himself definitively to the idea of EP's being composed only by names.

Wittgenstein's view - as we interpret it - that sentences are combinations of individual names only, does not hold for the sentences that are used in actual fact. Wittgenstein is aware that the language he outlines is an unrealized ideal. (KUNG 1967, 82)

In a note on page 81, Kung added a bibliographic review on those who adopt the same interpretation as him concerning the exclusion of relations and other properties from an ideal language. For him: "Wittgenstein's ideal language contains no designations of relations".

II.3 EPs AS SEQUENCES OF INTERCONNECTED NAMES AND THE IDEA OF AN INTERNAL RELATION

The second important feature emphasized by Wittgenstein concerning EPs' logical form is the idea of an internal relation among the referents of its constituent's simple names, the simple objects (or ingredients). The EPs' respective simple signs should be concatenated by some structural relation and result in a sequence of connected names that must mirror the connection which holds between their respective atomic ingredients. For Wittgenstein, there should be an "internal articulation" connecting each simple object (or ingredient) to the others, each simple name to the others as well. An EP must not be a random sequence of ingredients' names, put together in any order, but a structured and organized sequence.¹⁵

One essential aspect in the understanding of Wittgenstein's construal of an "internal relation" is the spatial arrangement between the elements of the "list". This pictorial arrangement, one element after the other in a line nearer to one than to another, named by him "Internal connection", should be seen as an aspect of the notation. It is an event in the world, but an event that cannot be figured by another meaningful proposition.

In the *Tractatus*, Wittgenstein had tried to explain this special kind of relation which must hold internally between the atomic elements.¹⁶ It was a persistent idea in Wittgenstein's writings that those simple atomic objects must have a very special kind of *internal* connection, which could be displayed by the disposition of their names in their written presentation.

4.122 We can speak in a certain sense of formal properties of objects and atomic facts, or of properties of the structure of facts, and in the same sense of formal relations and relations of structures. (Instead of property of the structure I also say, "internal property"; instead of "relation of structures", "internal relation".

I introduce these expressions in order to show the reason for the confusion, very widespread among philosophers, between internal relations and proper (external) relations.) (1961)

This "internal" relation should not be described in any discursive way but must just be *seen* from the sequential arrangement of the EP's syntactic elements, their constitutive signs, one in relation to the other.¹⁷

4.122 The holding of such internal properties and relations cannot, however, be asserted by propositions, but it shows itself in the propositions, which present the facts and deal with the objects in question. (1961)

4.1221 An internal property of a fact we also call a feature of this fact. (In the sense in which we speak of facial features.) (1961)

15 We will return to this point in the final sections of this paper.

16 In the paper (VELLOSO, Wittgenstein "Great Analysis" and Frege's construal of number as a property of properties 2017), I discussed Wittgenstein's construal of internal relation in connection with Frege's account of number as second order concepts. There, we compare Wittgenstein's notion of "internal relation" presented in the *Tractatus*, with his later notion of "aspect seeing" discussed in the (WITTGENSTEIN, Lectures on the Foundations of Mathematics 1939 1976).

17 At least not by what Wittgenstein had call propositions with sense in the *Tractatus*.

As we said, to safeguard the PLIEP, Wittgenstein assumed in the *Tractatus* that all predicates must be removed from the internal structure of EPs. For him, general terms should be derived afterward from the way those basic ingredients were organized inside the EP.

Let us return now to the other idea we want to discuss in this section, the idea of EPs as “lists”. As we discussed in the last two sections, Wittgenstein contemplated several times the idea of getting rid of all predicative expressions from an EP and just sticking with names arranged in a sequence. But how did he think the logical form of this EP should be? What could we give as an example of that? To begin answering this question, let us focus again on Wittgenstein’s concerns about the internal link between Atomic SoAs’ components and their respective names, together with their possible relationship as parts of the same whole, without trying to anticipate too much for now of the discussion about the nature of those ultimate factors.

In SRLF, he summarized his alternatives. According to him, there were only two alternatives: to cut off those predicates and get the final sequence consisting just of numerical coordinates, the genuine EPs, or to give up the goal of finding a logical product of propositions with independent truth values.

A simple example would be the representation of a patch P by the expression “[6-9, 3--8]” and of a proposition about it, e.g., P is red, by the symbol “[6-9, 3--8] R”, where “R” is yet an unanalyzed term (“6--9” and “3-8-” stand for the continuous interval between the respective numbers) (WITTGENSTEIN, 1993, p. 166) [my emphasis]

In this passage, Wittgenstein is calling our attention to this remaining residue of analysis: “‘R’ is yet an unanalyzed term”.¹⁸ In the first moment, still involved with the *Tractatus*’ writing, Wittgenstein had underestimated the task ahead of him, as he commented years later:

When I wrote the *Tractatus* (and later as well) I believed that [fa = fa & ~fb] would be possible only if fa were the logical product of some other proposition and ~fb – and therefore fa = p & ~fb – and I was of the opinion that fa (e.g., a colour-statement) could be analyzed into such a product. In this context, I had no clear idea about how I was imagining the discovery of such an analysis. (BT, 340e)

Thereof, at the time of the *Tractatus*, he had believed the finding of this logical product possible but did not stop to consider the details, as he puts it later on in *BT*. The whole tratarian idea of completing the GA involved reducing the predicate for color to a logical product of EPs, each one asserting a certain grade of each component of the final color (brightness, intensity, and chroma). Any as yet incompletely analyzed proposition, as *fa*, for example, should be re-written as *fa* = p & ~fb, where “p” should be a logical product of EPs. Right after this observation, he added the following remark about what he ended up concluding:

Then, when I wanted to carry out such an analysis of a colour statement, it became apparent what I had imagined analysis to be. I believed I could understand a colour statement as a logical product r & s & t..., the discrete factors of which indicated the ingredients (if there were several) that the colour (“colour”, not “pigment”) consisted of. (The Big Typescript TS 213 2005, 340-1)

18 In (VELLOSO, 2014), we argued for the absence of predicates within the EPs, as this passage seems also to point out when Wittgenstein says that: “‘R’ is yet an unanalyzed term”. We are now ready to advance one more step and try to investigate in detail how this claim would reverberate in the EPs’ logical form.

In this passage, we get another moment of retrospective clarity from Wittgenstein. He even visualized how the logical form of EPs should be. As we are going to explore in more detail in the next sections, Wittgenstein's suggestion was to understand each EP as a list of names asserting the occurrence of one of the "discrete factors of which indicated the ingredients (if there were several) that the colour ("colour", not "pigment") consisted of". Each list, moreover, would have to display numerical intervals needed to identify the location and the color. According to that explanation, EPs will consist of a final articulated list of numerical coordinates.¹⁹

Two lessons can be learned from our discussion until now. The first is that the EPs' ontological correlates, the atomic SoAs, should have parts. The second lesson is that the atomic SoA's parts are what Wittgenstein was calling "ingredients/elements" and should be labeled by "simple ingredients names". We could even say that at the *Tractatus* he had only a vague idea of what those ingredients were supposed to be. They would be instants of time, points in space, and, finally, color's ingredients ("color, not pigment", like bright, intensity, and chroma), as well as ingredients from the other sensorial spaces as well, like sounds, tastes, smells, cold and hot, etc. He later imagined them as coordinate numbers used to localize a colored point in the phenomenological space.

Our conclusion in this section is that the next difficulty we have to face remains in those residual color predicates. We will have to understand now Wittgenstein's effort to replace them with numbers at the final phase of GA. This final phase makes all the difference between Wittgenstein's method of analysis and that of Frege's. He, in the opposition to his master, thought that the level of singular propositions was not the *ultimate end* of the analytical process. He insisted that it was necessary to go further and to find the real building blocks of sense and truth.²⁰

III. THE COLOR INCOMPATIBILITY PROBLEM

III.1 WITTGENSTEIN'S DIFFICULTIES WITH THE FINAL PHASE OF THE GA

Resuming our conclusions from section II, Wittgenstein's suggestion in SRLF was to understand a draft version of the final genuine EPs as n-tuples of numerical coordinate names like: "[6-9, 3-8]", plus a color predicate "R". Further on, this last residual predicate asserting a degree of color to a location, or *contrariwise*, would have to be analyzed away as well.²¹

In this same paper, Wittgenstein finally offered a brief summary of how he thought the solution would be and explained the reasons why it was not achieved.

One might think – and I thought so not long ago – that a statement expressing a degree of quality could be analyzed into a *logical product of single statements of quantity* and a completing supplementary statement. As I could describe the contents of my pocket by saying "It contains a penny, a schilling, two keys, and nothing else".

[...] But this will not do as an analysis of a statement of degree. For let us call the unit of,

19 We are going to explore in more detail the idea of "list" in section IV. For now, we need only to say that an EP should be composed of signs disposed in line, one after the other, from left to right.

20 Cf. Velloso, "Frege and Wittgenstein's debate regarding the notion of 'fact' in the *Tractatus*", forthcoming, p. 17-18.

21 In manuscript 105, (WITTGENSTEIN, Wiener Ausgabe Studien Texte (Band 1 - 5) 2001, 15, Band 1, §§2-3), Wittgenstein defines color and space as interpenetrating each other, i.e., there is no logical priority between them, so that one should be treated as the bearer and the other the predicate, or vice-versa: "It is clear that there is no relation of 'being at' between a color and a place in which it 'is'! There is no link between color and space. Color and space saturate each other."

say, brightness b and let $E(b)$ be the statement that the entity E possesses this brightness, then the proposition $E(2b)$, which says that E has two degrees of brightness, should be analyzable into the logical product $E(b) \& E(b)$, but this is equal to $E(b)$; (pp. 32-33, SRLF)

In this passage, Wittgenstein explains what was for him the best solution to the color incompatibility problem: to achieve a final logical product of EPs, exemplified by $E(b) \& E(b)$, or “[6-9, 3-8](b) & [6-9, 3-8]b”. According to him, “E” would be a numerical coordinate naming a definite location in the visual field and “(b)”, the name of “one unit of brightness” that could be said to “be in” this location.²²

This suggestion of construing an EP as a series of numbers’ intervals, however, involved asserting each occurrence of a unit of brightness separately.²³ Each one would be asserted by a single EP. Only after asserting each one in isolation, the propositional operation of conjugation should deliver the final result: a total number of grades of brightness in a certain location. This final product should be the result of finally analyzing away the residual predicate ‘R’.

In the preceding passage, Wittgenstein also explained what should be for him the more significant and insoluble problem blocking the achievement of this final product. He comments that the replacement of “ $E(2b)$ ” by two atomic EPs, “ $E(b) \& E(b)$ ”, would be equivalent to the assertion of the same first proposition once. The trouble is that in the propositional logic the double assertion “ $E(b) \& E(b)$ ” is equivalent to one assertion of $E(b)$ and, in this way, one could not obtain the desired additive result, “ $(E)2b$ ”.

In the *Manuscripts*, he also addresses this same problem and quotes Frege regarding this trouble:

1. * [In the logical theory of color it can be good to remember the difference between $2 + 2 = 4$ and $\phi a \cdot \phi a = \phi a$ [Frege] (2001, 63, Band 2)

After rejecting the idea of asserting the occurrence of one unit of bright twice, Wittgenstein considered another solution that still involved the idea of a “logical product”. He suggested making a notational distinction between the two basic propositions “ $E(b) \& E(b)$ ” by adding prime symbols to the “b” signs. This new suggestion involved introducing one more prime to the superscript right side of the letter “b” to distinguish those two possible assignments. It would be one prime symbol for one single unit of brightness, two primes for the assertion of two units of brightness, and so on. With this device, Wittgenstein expected to preserve the idea that a logical product of EPs would set up the truth-condition for all the other meaningful propositions. The result should be the right intensity of brightness in each singular location, something like $E(b')$ & $E(b'')$, as he says in the next passage:

[...] if, on the other hand, we try to distinguish between the units and consequently write $E(2b) = E(b') \& E(b'')$, we assume two different units of brightness; and then, if an entity possesses one unit, the question could arise, which of the two – b' or b'' – it is; which is obviously absurd. (Some remarks on logical form, pp. 32-33)

The importance of presenting this solution is to show that Wittgenstein needed to operate with those units and obtain the total amount of color ingredients located at some point.

22 But things could be the other way around, as he puts in M105, p.15, 2-3, “It is clear that there is no relation of ‘being’ between a color and a place in which it ‘is’! There is no link between color and space. Color and space saturate each other.”

23 Nested, because it would contain a multiplicity of numbers’ sequences.

As we can see from this last attempt, he was clearly regretting the idea of relinquishing the truth-functional character of this operation, for he has still tried to understand it as the result of a logical product.

According to Wittgenstein however, this solution turned out to make things even worse than they were before. This second reductive form would assert the absurd situation of having two distinguished simple signs denoting different grades of brightness intensity: b' or b'' . In this awkward situation one would have but to choose between those two different assertions. Wittgenstein finally realizes that instead of adding two units of brightness to obtain the double occurrence of one unit he would still have something contradictory: the assertion that two different degrees of color's brightness qualified the same location.

Wittgenstein's idea of simply putting one prime representing each factor (ingredient, or unit of bright) seems to imply that one could see those prime symbols as one single aggregated whole composed of a multitude of *discrete or independent* units.²⁴ In our opinion, all those experiments arise from a single difficulty: how could one apply a propositional operation to EPs and get the sum of each factor? As we construe the situation, Wittgenstein has concluded that one must first be able to represent numbers as a sequence of signs and just then apply some kind of additive operation, to finally be able to ascribe the right total number of factors' colors to a location. Wittgenstein's later conclusion was that the truth-functional character of propositional logic would not be able to perform such a task. He then thought that what he really needed was to apply the sum only to the internal elements of the EP.

In the following passage, Wittgenstein confirms his relinquishing of the idea of extending "truth-functional" analysis to the ultimate level of the GA.

1. / This makes it appear as if construction [*Konstruktion*] could be possible within the elementary proposition. That is as if there was a logical construction that *does not work with the help of the truth functions*. [...]
2. That's what I wanted to say with my relations that are expressed by numbers. [my emphasis] (2001, 56, Band 2)

In this passage, he is giving up altogether the very idea of *vero*-functionality and looking for an alternative logical construction within the EPs. According to this novel approach, the process of incrementing a further unit to the last obtained total would be an internal procedure taking place inside the EPs and not a truth-functional operation acting over propositions.

In the next sub-section, we are going to explore this alternative solution as it was investigated by Wittgenstein in the *Manuscripts*. This novel approach aims to show how one would get a total number from a sequence of units by an internal procedure happening inside de EPs.

III.2 NUMBERS AS SCHEMES

The idea of a "logical product of propositions" was the first sketch of a solution for the color incompatibility problem investigated by Wittgenstein in the *Manuscripts* 105-106. His second idea to add those units and obtain the final degree of color was to get rid of concepts and operate directly upon the component elements of each EP. According to him, this second approach involved still considering numbers as an assertion about the extension of the concept,

24 In the two previous quotes, Wittgenstein used the two words "units" and "factors" interchangeably. We think that at both times he meant by those words the same thing: the isolated units of each group, brightness, intensity and chroma, which compose each color.

but not as a second order assertion, as we are going to see next. For this purpose, he introduced in the *Manuscript 105*, the idea of a “scheme”.

8. The number is a scheme.
9. When I speak of the “number of books on this table” I mean a certain scheme of the kind ||||... or 0, ξ , $\xi+1$, which can be applied to the range of this concept. (2001, 26, Band 1)

After presenting his construal of “numbers” as “schemes for concepts’ ranges”, he explored the idea of dropping out the conceptual, or intensional part, and operating only with the extensions:

4. / One could ask: does the number have essentially something to do with a concept? I believe this boils down to asking whether there is any sense in talking about a number of objects that are *not* grouped together under a concept. For example, does it mean something to say: “a and b and c are 3 objects (*Gegenstände*)”? I believe, very often, not! However, there is a feeling that tells us: why talk about concepts; the number depends only on the range of the concept and once that has been determined, the concept can, so to speak, be dropped-out. [...] In the opposite case, however, the range independent of the concept is only a chimera and then it is better not to speak of it at all but only of the concept. (2001, 30, §4, Band 1.)

It is a fact that in this passage Wittgenstein seems to be wavering between two opposite conceptions: the extensional point of view, as he called it, according to which one should drop out the concepts and work only with their extensions; and the intensional point of view, according to which concepts were essential for the understanding and manipulation of their extensions and should not be dropped out. Regarding what comes immediately following this passage, we believe that he tried the former approach in the *Manuscripts*, the extensional option, although with some doubts and uncertainties. His goal was to see if with this approach he could properly deal with the most basic arithmetic operations by considering them as internal operations that happen inside the EPs.

8. The number sign is a scheme and is torn out of its context in arithmetic. (Ms-105,115[2]_1, 1929-02-02?-1929-03-31?)” (Wiener Ausgabe Studien Texte (Band 1 - 5), 29, Band 1).

In this sequential passage he emphasized the idea that a “scheme” did not need to involve a propositional context. His idea was to deal just with the signs themselves, providing a special notation that could display all the components units of the final number. In M106, we could find several passages where Wittgenstein was trying to operate only with schemes, as we are going to see next.

Wittgenstein’s temptation of decomposing numbers into units connected like the links of a “chain”, was still an effort to achieve the last level of reduction required by the GA, although in the internal context of each EP. The treatment of predicates for colors’ degree would be achieved through the postulation of this internal procedure that should involve seeing the “perceived color’s different mixing ratios” as additive series of units connected inside an EP (2001, 22, §4, Band 1).

This time, Wittgenstein was completely relying on that other idea of an internal relation which was supposed to indicate the final degree of color. This internal relation was explained by him with the idea of the notation which must display the sequence of signs in a certain specific arrangement. He also adds to this idea another property of this notation: that it could, at the same time, denote and exemplify the summands of a number (we will call it, from now on, “*the bar notation*”).

8. / It seems to me (Namely) that the decomposition of a number into its Summands is an immediately illuminating operation and does not need an introduction by way of operations with truth functions.

So, it seems to me that one could say straight away, "You see, |||| consists of || and ||." (2001, 31, Band 2)

According to Wittgenstein in those last passages, an internal relation is something very distinct from a truth-functional relation. He pointed out that the items displayed by the bar notation should be connected in the same way as the links of a chain, i.e., without an external third element that carries out the connection from one to the other. In the following passage, he calls this idea of an "internal relation" between the "bars" an "addition".²⁵

5. / As I said, it is clear that the sentence that a color contains 5 stitches of yellow cannot say it contains stitch No. 1 and it contains No. 2, etc. but the addition of the stitches must take place within the elementary sentence. But how if these stitches were objects (Gegenstände) that are in a certain way line up like links in a chain and one sentence speaks of five such links, in another sentence of three. [my emphasis and translation] (2001, 60, Band 2)

In this passage, he goes directly to the heart of the matter. The unanalyzable predicate "R" would be the result of an internal operation taking place inside each EP, which should add those units or *stitches of yellow*. He also says the connections between those bar units should be regarded as an essential property of their disposition in the scheme. In those other passages he comments again upon the requirements of his GA:

5. / [...] Yes, but these two propositions must exclude each other without being decomposable. (2001, 60, Band 2)

1. * If something is wrong with my foundations, it could only be that there are essentially no elementary sentences at all, and that the analysis yields a system of propositions decomposable to infinity. Doesn't this system meet the requirement of the specificity of the analysis that I am making? (2001, 50, Band 2)

Of course, this would mean a drastic change concerning his approach in the *Tractatus*. To begin with, it would be the end of the ultimate "building blocks idea". At the final phase of the GA, one should simply get a propositional level of not completely analyzed "EPs" that should be able to "be broken down to infinity".²⁶ Any further process happening after that should take place within each EP. This inside procedure thought should be more like a kind of "recognition process" than an analytical one, for we will no longer be replacing one of the proposition's terms with other equivalents. In this other passage, remarkably similar to those we quoted from SRLF, he repeats his conclusion that he cannot proceed with the process of replacing these residual predicates for other simpler terms:

3. / [...] And different degrees of red are incompatible with each other. One could think of it as explained in the following way, that any small quantities of red add up to a certain degree of red. But then what does it mean to say that there are about 5 such quantities of

25 He uses more than one name for the signs in the scheme. In this passage he has called them "stitches" in connection with color's points. Wittgenstein's intention, thought, is to avoid any systematic way to name those formal structures, as we are going to explain later.

26 Although Wittgenstein is giving up the terminology of Elementary Proposition, we will still apply this category to distinguish the singular elements from the other propositional elements of the language.

red? Of course, this cannot be a logical product, that there is quantity No 1 and quantity No 2 to 5, because how would these differ from one another? So, the proposition that the degree 5 of red is present cannot be decomposed in this way. [my emphasis] (2001, 55, Band 2)

The decision of introducing numbers for degrees of color into the internal structure of the EPs was actually a very radical move from Wittgenstein. As we have discussed at the beginning of this paper, the scenario of foundations of mathematics was at that time in a very chaotic situation, mainly as a result of Russell's paradox. So, his efforts to understand numbers and the final ground level of language were completely comprehensible. In the next section, we intend to go back to Frege and his discussions regarding "aggregative thinking" in mathematics. We believe that the main interlocutor and opponent in Wittgenstein's reflections from the manuscripts was Frege's *Grundlagen* discussions concerning the notion of "unit".

III.3 "THE EMPTY FORM OF THE DIFFERENCE".

In *Grundlagen* (§36) and in *Grundgesetze* (vol II, Part III, §151) Frege presented long arguments against the idea of conceiving numbers as the result of joining or counting units of any sort. Frege's problem was not with counting things in general but with counting units, even the abstract or merely syntactic ones. In fact, his discussion also criticizes the idea of "a unit" *in abstract*, as his quoting of Leibniz's dictum "*Abstractatum ab uno est unitas*" shows.²⁷ (FREGE, *The Foundations of Arithmetic* 1953, 48, §37)

During the debate with his fellows, Frege gave many reasons against their treatment of numbers as *sums or wholes*, which would be the result of putting together, or counting, identical, but nevertheless distinguishable abstract units. As he accuses Leibniz, the error involves taking a concept as indistinguishable from its instances, i.e., of confusing concepts and objects. Frege's accusation against Leibniz was in accordance with his own third principle, presented at the end of *Grundlagen*'s introduction: "never to lose sight of the distinction between concept and object." Frege's main concern was with the use of the notion of "unit" in those formulations. In his opinion, this idea of "unit", which should be at the same time a concept and its instance, is itself absurd. He complains that this kind of object must be a singular entity, to represent the number one, but at the same time, it must represent all other units, or "ones", identical to it.²⁸

Frege explained what he meant as follows. To obtain any number different than one, i.e., to form a number by joining isolated units, one must be successful in distinguishing those units from one another, otherwise, all those component's units would collapse into a single amorphous whole. The outcome in this last case would be one single unit identical to itself. But the exact moment one obliterates the distinction between them is the moment when those units cease to be different things. Or, to put in other words, on one hand, counting units is paradoxical, or circular, because the concept of number must be already presupposed. But, on the other hand, taking the units as the number itself leads us to the absurd situation of their

27 Frege interpreted this quote as saying that: "Leibniz understands by *unitas* a concept under which this one and that one and the other one fall, or as he also puts '*Abstractatum ab uno est unitas*'". He was criticizing the lack of a clear distinction between the concept and the object that falls under it.

28 One proviso has to be made here, Frege never thought of regarding the signs used in a notation as anything more than mere arbitrary signs. So, his arguments were always intended against, either the conception of numbers as abstract objects, or else of numbers as just signs without content. We think that Wittgenstein was trying to argue with Frege that there might be a formalist but interesting third option. However, that is not our main objective here. Therefore, we will leave this point to be developed in another paper.

indistinguishability, for then a question would arise: which "one" would be the first "one", and the second "one", etc.? Or, else, how could we sum them all and achieve something different from a series of "ones"?

Frege concluded that the very idea of "unit", either considered as an abstract object or else as a notational device, was absurd. As he wrote in the *Grundlagen*, page 46: "exact identity is unity, and with difference arises plurality." On page 56, he spoke again against the idea of conceiving numbers as compounds of units of *any kind*: "Abstract number, then, would be [wrongly taken as] the empty form of difference."

To help clarify the matter, Frege gave us as an example a passage from the work of Jevons:

Jevons goes on: "Whenever I use the symbol 5 I really mean

$I + I + I + I + I$,

and it is perfectly understood that each of these units is distinct from each other. If requisite, I might mark them thus

$I' + I'' + I''' + I'''' + I'''''$."

Certainly, it is requisite to mark them differently, if they are different: otherwise, the utmost confusion must result. (FREGE, *The Foundations of Arithmetic* 1953, 47, §35)²⁹

Immediately after this passage, Frege complains that Jevons' expression "ones" would be misleading and indicates the root of the difficulty. For him, there would be only two options at Jevons' disposal. Either the difference is only in the symbols, the strokes, bars, or any other arbitrary sign, or else this difference is in the referents.³⁰ In the first case, we would have just a terminological difference in the signs employed. In the second case, the difference would be objective, i.e., in the things themselves. After exposing this idea, he presses his adversary further with the following question: if those agglomerated units must be different from each other, would we have to re-write our equation as: " $a + b + c + d + e$ "? Frege's question is very discerning because when one confronts herself with letters, adding those letters suddenly appears to be an absurd proposal. He then summarizes his conclusion:

So, our one slips through our fingers; we are left with the objects in all their particularity. The symbols I' , I'' , I''' tell the tale of our embarrassment. We must have identity hence the I ; but we must have a difference – hence the strokes; only, unfortunately, the latter undo the work of the former. (FREGE, *The Foundations of Arithmetic* 1953, 47-48, §35)

Frege's own approach is well-known. The cardinal numbers must be formulated using the property of "equinumerosity". This second-order property was for him a property applicable to concepts and not to the objects which fall under it. To bring those second-order properties to level zero, Frege introduced in the *Grundgesetze* a cluster of extensionality operations, as we have already presented in subsection I.1. In addition, cardinal numbers should be identified with those extensions and should be considered as zero-level entities. Finally, ordinal numbers would be derived from his six logical axioms, including the successor function.³¹

We are not arguing in this section that Wittgenstein had adopted Mill's purely empiricist ideas, so despised by Frege. He would never confuse each unit with a concrete body, or even

29 Ver em FA, Jevons *apud* Frege em nota: "op. Cit. p. 162 [1874 end., p. 182]".

30 For Frege, there were only those two options available and no intermediary way to follow between them, as we said in footnote 28.

31 Those axioms are equivalent to Dedekind-Peano axioms.

with, say, an atom. Quite the contrary, in the Austrian philosopher's account, one could observe at least two important aspects, which would make his solution a very feasible option. The ideas of "the increasing by one operation" and of "displaying the scheme". Those ideas were also emphasized by Frege in his accounts of Kant's and Leibniz's construals of numbers, as we are going to see next.

IV- WITTGENSTEIN'S CONSTRUAL OF "NUMBERS" AS "SCHEMES", AN ALTERNATIVE WAY TO SOLVE THE COLOR INCOMPATIBILITY PROBLEM

IV-1 THE "INCREASING BY ONE" OPERATION AND THE "DISPLAYING" IDEA, KANT'S AND LEIBNIZ'S EXPLANATIONS OF NUMBERS

We believe that Frege's construal of "numbers" as "second-order properties" and his arguments against considering them, either as objects, or as an abstraction from objects of any level, or else as first-order concepts, made an extraordinarily strong impression on Wittgenstein. As we have said at the beginning of the paper, Wittgenstein's misgivings regarding having numbers to designate degrees of colors had to do with his admiration for Frege's ideas, on one side, and with his vigorous desire to fix Frege's extensionalist definition of numbers and avoid Russell's paradox, on the other.

As we discussed in sections III.1 and III.2, something went astray with the idea of reducing colors to a logical product of propositions. We have then suggested that we could find in *Manuscripts* 105 and 106 an alternative solution, not presented in SRLF. Wittgenstein's proposal this time would be to understand "numbers" as "schemes", and Addition as an internal operation occurring inside the EPs. They would be ideal structures composed of bars or any other signs that stand for the units which make up the number.

The problems faced by Wittgenstein in those *Manuscripts* guards similarities with some of Frege's opponents' views about numbers discussed in the *Grundlagen*. We would even risk saying that Wittgenstein's proposal reveals an attentive reading of Frege's book and entertains a very akin nature to Kant's proposal, as well as, in some respects, also to that of Leibniz. The similarity with the latter reveals itself through the idea of composing the series of natural numbers by the "increasing by one operation". The similarity with the former comes from the idea of a visual scheme composed of units construed as bars, points, or fingers, as in Kant's famous passage (KANT 1998, B15-16), but also, in some sense, from the "incremental idea" present in that passage.

The similarity between those three accounts regarding the nature of numbers comes, in our opinion, from those two central aspects of Wittgenstein's idea of "scheme": (1) the appeal to a visual kind of proof, provided by a notational device of bars in which the bars denote each unit, but also instantiate each one of them; and (2) the idea of obtaining the series of natural numbers by the incremental operation. In Wittgenstein's construal, these two characteristics were exhaustively discussed. His schemes should at the same time display the summand units of a number and instantiate them as links of the same chain. Wittgenstein's idea was to see their internal connection as a recursive procedure of increasing the previous total by each one of the sequential units.

The resource to a notational device that both denotes and exemplifies those units, though, is an obliteration of the boundaries between concepts and objects according to Frege's view. As he warned many times, "one should not lose sight of the distinctions between concepts and

objects". In fact, it is not just a matter of *losing or not sight* of a philosophical distinction. We believe Wittgenstein was fully aware of the implications entailed by his proposal. His suggestion actually involved a *deliberated* and careful union of those two aspects of the idea of "unit", the conceptual and the objectual one, into one single notation, the *bar notation*. Let us analyze those two aspects in detail.

In those famous passages from Kant, the idea of a visual scheme plays a significant role as can be seen by his analogy of using fingers and putting them together to form the concept of twelve:

The concept of twelve is by no means already thought of merely by my thinking of that unification of seven and five [...] One must go beyond these concepts, seeking assistance in the intuition that corresponds to one of the two, one's five fingers, say, or five points and one after another add the units of the five given in the intuition to the concept of seven. (KANT 1998, 144, B15-16)

Mathematics fulfills this requirement by means of the construction of the figure, which is an appearance present to the senses (even though brought about a priori). In the same science, the concept of magnitude seeks its standing and sense in number, but seeks this in turn in the fingers, in the beads of an abacus, or in strokes and points that are placed before the eyes. (KANT 1998, 341, A 240)

In these passages we can find those two elements employed by Wittgenstein in his idea of a "scheme": the increasing of units "one by one" and the idea of "seeing" those units in a figure with the help of its representants: "fingers, beads of an abacus, strokes or points".

As in the case of Kant's explanation (1998, 341, A 240), Wittgenstein's "scheme" was also a purely visual structure, which could be applied to empirical situations of counting. Wittgenstein added one more characteristic to this ideal structure, though. He said that what must be counted was not the bars themselves, but what those bars "have in common with all classes of four, which I can't represent without one [scheme]." (2001, 26, §§8-9, Band 1)

Wittgenstein's scheme only displays the bars segregated into groups. As we are going to see in section IV, some process of aggregation, or incrementation, should be implemented in order to obtain the final total number. Something similar happens in Kant's explanation quoted above: "and one after another add the units of the five given in the intuition to the concept of seven".

It is regarding this very point that we encounter a resemblance with the idea of an "increase by one operation" suggested and discussed by Frege here.

Every number, that means, is to be defined in terms of its predecessor. And actually, I do not see how a number like 437986 could be given to us more aptly than in the way LEIBNIZ does it. Even without having any idea of it, we get it by this means at our disposal nonetheless. Through such definitions we reduce the whole infinite set of numbers to the number one and increase by one, and every one of the infinitely many numerical, formulae can be proved from a few general propositions.

Now we have already decided in favor of the view that the individual numbers are best derived, in the way proposed by Leibniz, Mill, H. Grassmann, and others, from the number one together with an *increase by one*, but these definitions remain incomplete so long as the number one and increase by one are themselves undefined. [my emphasis] (FREGE, The Foundations of Arithmetic 1953, 25, §18)

In this passage, Frege presented the idea of an "increasing by one operation" and of constructing the ordered series of natural numbers starting from the already defined number one. He points

out, though, that the “incremental idea” taken as an operation that generates the natural numbers was a much more sensible device to deal with large numbers than the simple idea of counting.³² In fact, he says that he could not see any other way to reach a number that is not intuitable.

Despite being comfortable with the idea of “the incremental operation” and realizing that it is the right and sole way to achieve a complete generality in the process of defining the series of natural numbers, Frege warned us again. This time he was concerned with the temptation of relying too much on this “showing” idea. According to him, one should not consider the display of the serial addition as a kind of “visual proof”. His advice was that one should go for the harder task of defining the “number one”, as well as the “incremental operation”.

As we’ve said, Kant also recommended some kind of incremental operation without any previous definition. For him, it was a knowledge of an intuitive kind: “I now add the units that I have previously taken together in order”. Frege presented and criticized those views in *Grundlagen*:

In this way we get a symbol such as: $I + I + I + I + I$, the composition of which we can describe by saying: ‘A natural number is a sum of ones’.

This passage shows that for SCHRODER number is a symbol. What the symbol expresses, which is what I have been calling number, is taken, with the words “how many of such units are present”, as already known. Even by the word “one” he understands the symbol I , not its meaning. The symbol $+$ is introduced solely to serve as a visible mark, without any content of its own, for linking up the other symbols; only later does he define addition. He could indeed have put what he means more briefly by saying that we write down, side by side, as many symbols 1 as we have objects to be numbered and link them up by the symbol $+$. Naught would be expressed by writing down nothing. (1953, 55, §43)

In this passage, Frege discusses the situation in remarkably similar terms to those we find in Wittgenstein’s *Manuscripts*. The similarity though is just superficial. Contrary to Wittgenstein, Frege has resisted the idea that a pure concatenation of signs could *generate* a number. He complained many times that according to this *aggregative idea* a number would be a sign without content. Despite Frege’s criticisms, Wittgenstein insisted on applying the incremental operation to those units (represented by the bars) which compose the scheme.³³

We believe that Wittgenstein was fully aware of Frege’s advice. Despite this profound respect for Frege, he suggests in the *Manuscripts* that the problem of defining one and the incremental operation could be solved by the *bar notation*. Apparently, he was thinking that this last kind of notation could combine both ideas without requiring any additional linguistic explanation.

On another side, we can find passages from *Grundlagen* where Frege discusses the “putting together” idea with complete skepticism.

When Hankel speaks of our thinking or putting a thing once or twice or three times, this too seems to be an attempt to combine in the things to be numbered distinguishability with identity. But it is obvious too at once that it is not successful; for his ideas or intuitions of the same object must, if they are not to coalesce into one, be different in some way or other. I also think that one is entitled to speak of 45 million Germans without first having thought or posited a normal German 45 million times; that would be a bit laborious. [my emphasis] (FREGE, The Foundations of Arithmetic 1953, 54, §42)

32 “Now this can only mean that from the way in which a number, say 8, is generated through increasing by one all its properties can be deduced.” (FREGE, The Foundations of Arithmetic 1953, 16, §10)

33 He even tried to treat the “+” sign as a comma, as we can see in: (WITTGENSTEIN, Wiener Ausgabe Studien Texte (Band 1 - 5) 2001, 42, Band 2)

For Frege, "never to confuse concepts with the objects that fall under it", was a very firm principle, connected to his most engrained philosophical ideas. It should be observed, especially in these cases when those objects are pure schematic units, deprived of any content of their own.

As we said before, the "scheme" is an idealistic structure, more akin to Kant's synthetical *a priori* schematism, or even to Leibniz's construal of an "*Abstractatum ab uno est unitas*". But, as Frege remarked in the last passage above, even an idealist explanation of numbers involves the idea of putting together identical, although distinguishable units and, so, displaying "*the empty form of the difference*". Let us now explore a little bit how Wittgenstein has tried to deal with all those various difficulties.

IV-2 THE OCTAHEDRON AND THE IDEA OF A TOTAL MIXTURE

Wittgenstein's arguments in support of the construal of numbers for degrees of colors as schemes, which display an internal relation between the numbers' component units, appeared in detail only in the *Manuscripts* 105-106. The first novelty concerning the color incompatibility problem found in those texts is that each atomic and indivisible-colored point could be indicated by the assignment of a mixture of numbers. This mixture should describe each color completely.

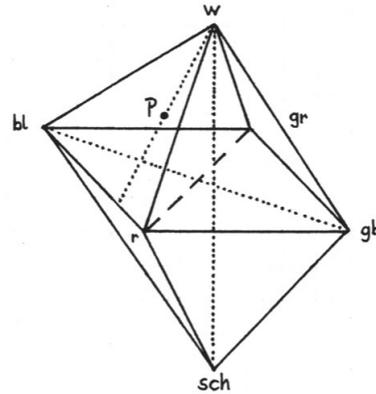
1. /Could not the following scheme exist: The color in a point is not determined by the assignment of a number to a point, but by assigning several numbers. A mixture of these numbers makes the color and to describe the complete color I need the sentence [saying] that this mixture is now the complete mixture, so it can no longer be added. It would be like describing the taste of a dish by listing or enumerating (aufzähle) the ingredients; (2001, 21, Band 1)

According to Wittgenstein in this passage, the assignment of several numbers to a point would demand that we figure out which mixture of ingredients would result in that exact degree of color.

1. [...] then I have to add at the end that these are all the ingredients. [...] You could say that the color is only then fully described if all their ingredients are indicated, of course with the Addition that it is all. (2001, 21, Band 1)

The expression: "*these are all the ingredients*" in the passage has exactly the role of an extreme clause. For Wittgenstein, the amount of each ingredient and the extreme clause were both essential conditions for the determination of the exact degree of a color. So, regarding the idea of a mixture of colors happening inside each EP, the real contradiction would not be in the predicate itself, but in the extreme clause, which should determine an exact amount of each ingredient that forms the resulting color and preclude any other mixture. Nonetheless, the amount of each ingredient was an essential element in the determination of the color and must be understood in terms of quantities. This new idea consists in taking the octahedron's whole structure, including its internal connections, as a tridimensional diagrammatic schematization of the color' space. One should then consider "the spatial relation between two [colored] spots [represented in this diagram] by a and b, [...] as "a N b", where N is a number and thus an elastic relation". (2001, 26, Band I.).

3. Every point on the surface of the octahedron represents a color, e.g., P is a whitish blue red point, which is closer to red than to blue. (2001, 26, Band 1)



After providing this initial explanation, Wittgenstein turned his attention to numbers and the bar notation as an alternative way to recover the exact amount of each ingredient of the color determined from inside each EP, as we have already discussed in section III.2.

IV.3 THE BAR NOTATION AS A WAY TO MAKE SENSE OF THE IDEA OF “UNITS AS THE EMPTY FORM OF THE DIFFERENCE”.

Wittgenstein’s next step to consolidate his proposal was meeting the challenge of obtaining the total mixture of a color. It was at this point that we said he was forced to face the problem of adding those bars and to see them as a whole definite number.

At that time Wittgenstein entertained the hypothesis that an EP could be such a schematic list and, so, could be composed of numbers’ intervals disposed of in sequence. That is to say, each EP should contain multiple schemes, each one naming a number so that each interval regarding a location, a time, and the ingredients of a color would be the result of a sum inside each scheme. According to the diagrammatic notation adopted to build the schemes (presented in section III.2 above) “a number could be decomposed in its summands” (2001, 31, §8, Band 1). Then, each bar of the list would stand for one unit of the resulting number.

As we said before, the bar notation is not an entirely problem-free strategy for introducing numbers. This device brings back some of Frege’s arguments against his opponents and their construal of numbers.³⁴ Wittgenstein’s idea of identifying numbers with this more abstract structure, the “scheme”, should face all those very arguments. One of them is how one could incorporate the idea of an “*incremental* operation” into the idea of “putting bars together in a list”. Or, in other words, how would Wittgenstein avoid the collapse of all those units into an amorphous aggregation, as Frege suggested. As we will see next, the implementation of his notational device would require something more than the pure listing of signs.

Wittgenstein has discussed several possibilities for construing the *incremental* operation in the *Manuscripts*. The first one was as an iterative aggregation of bars intercalated by comas, or of “ones” intercalated by the “+” (plus sign).

4. x [...] the number is understood as the sum of units or as a scheme in which the units are not connected by addition. So as 1 + 1 + 1 + 1 ... or |||| Or is the “+” just like a comma? I would like that it was immediately obvious that any group in 1 + 1 + 1 + 1 + 1 can be seen as (*auffassen als*) a number. [my emphasis] (2001, 41, Band 2)

34 As we discussed in section III.

In this passage, we can circumscribe two diverse kinds of operations: (1) addition; and (2) a scheme of *ones* or bars, "linked" together in a figure that should be seen as a number. On the same page, Wittgenstein seems to be comparing those two explanations: addition as an operation upon "ones" and the "aspect seeing" idea applied to the bars. The second alternative though implies the possibility of a visual recognition or the grasping of those sequences of bars, side by side, *as* a number.

We believe that for Wittgenstein the second kind of operation is closer to his idea of just displaying the internal properties of the scheme without the use of any external explanation.³⁵ In the way we are proposing to reconstruct Wittgenstein's distinction between internal and external properties, the external ones should involve linguistic explanations of the signs themselves. Those explanations would be needed in order to explain to someone how to follow the scheme and thus build up the number series. If these explanations are required, the use of formal concepts and meta-linguistic explanations would be inevitable.

In the last passage, Wittgenstein seems to be investigating which of those two alternatives could better explain the idea of seeing the "bars", or "ones", as the units which add up to yield the number. He was indeed highly ambivalent between the simple display of his bar notation and the idea of giving some sort of rule from which to build the series of numbers until the obtainment of the final total.

Just one page after having presented those two options, he addresses the problem again. This time, he puts the *incremental operation* in focus:

1.x If the number is *generated* by addition, a structure actually would arise: (((1) + 1) + 1) + 1) + 1 etc. and so an expression that should have parentheses. (2001, 42, Band 2)

In this passage, a structured group of additions should be generated with the help of parentheses. This kind of procedure involves a recursion, each new unit should be added to the preceding total, just as Frege discussed when he analyzed Leibnitz's definition (1953, 7, §6). As we can see in the picture, each operational step is displayed enclosed in parentheses before we go to the next one. This first construal of the incremental operation would be reminiscent of a flight of stairs so that each step clearly demands the use of parentheses. This formulation seems to imply also that each previous step is preserved by the notation, before going on and adding the next unit to the previous sequence. Wittgenstein was not happy with the usage of parentheses, however.

1. [...] But that is not what I want. An expression like "|||||" must not require any parentheses at all. Then, however, it [the series] (1, -, -1) is actually describable purely externally by an operation. Because when the operation "-1", i.e., the joint of a new unit, ought to make sense in itself, then the new unit must be added to the whole already existing number (Zahl), and then, we have just one expression in the parentheses. [my emphasis] (2001, 42, Band 2)

In this passage, we have an alternative way of seeing the bars, and of putting them together as well. It involved the generation of a series, the series "(1, -, -1)", by an *incremental procedure* given externally through the operation "-1". The advantage of this second explanation is that it expresses what Wittgenstein needed with the help of just *one pair of parentheses*. It would

35 A lot of questions could arise about this point, as for example: how those bars can be counted, if one has a lot of them? They must also be put in a correlation 1-1 with all the other classes of *n* elements. We are not going to comment on all those details here, because it is not our main concern.

not be the same as displaying the whole series of additions, one by one, or in groups isolated by parentheses. It would be a kind of “summarized recipe” of *how one should go, ahead ad infinitum*. The problem is that it would be an external procedure that must be explained to someone, and not an immediate recognition that would be obtained from the direct observation of the internal properties of a scheme and could be grasped just by looking at it.

Let us now try to explain how Wittgenstein has conceived this last construal of “addition”. Each recursive step of affixing an isolated unit to the rest would be meant as a singular procedure, which adds another unit to the previous series. In this case, the rule to execute the whole recursive procedure would be expressed by an external procedure of adding a new bar to the previous value of the variable and is described by the notation inside the pair of parentheses. This time, he was just giving a general “method” that anyone could follow to continue the series. First, take the first element, then put it in the inductive variable place (-), and then increase the value of this variable place by one more unit. It’s the same idea he used in the *Tractatus* to define numbers. But there, those formal series were the result of applying an operation to propositions, and, in the Manuscripts, they were used to iterate “ones” or “bars”.

Next, in the same paragraph, Wittgenstein compares those two conceptions with a third one:

1. [...] But here there is still another conception: the operation shows the relation from III to IIII etc. it goes from one to the other and this internal relation certainly has meaning. Well; only then the operation is not simply that of addition. (2001, 42, Band 2)

In this third case, the visual arrangement of the bars in two groups with a space between them would be the aspect which should allow us to see them as a sequence. Just as he had wished right from the beginning. But it surely presupposes that a correlation of one-to-one between the units of those two groups could be implemented. The expected result is that one must see the second group as equal to the precedent, except for the extra unit.

According to Wittgenstein, this last result should spring from the way the schemes were spatially arranged and from the presence of an extra bar in the second group. The resulting figure plays the role of justifying the passage from the first group of bars to the second through the displaying of the intuitive increase of one more bar. The incremental relation should then have become evident by the arrangement itself. This is what we believe he meant with the expression: “only then the operation is not simply that of addition”. It seems that one more property is being shown: that of “being a series in which the next member has one more bar than the precedent”, or else “being the series of natural numbers”.

In the next passage, Wittgenstein comments on the two initial methods: the one involving groups of ones separated by pairs of parentheses and the other involving a sequence of “ones” joined by the plus sign. Only in the first case, the recursive steps of putting the next unit together with the previous serial addition were shown separately with the help of parentheses or commas.

1. [...] Think about the generated sequence: 1, ((1)+ 1), (((1)+ 1)+ 1), etc. and then omit the parentheses. But that does not mean anything either, because where the sign “+” appears, it must be added to one number (Zahl). And that would mean that there cannot be an amorphous sign 1 + 1 + 1 + 1 as I thought; in other words, the “+” sign is not allowed to be used as a comma. (2001, 42, Band 2)

In this passage, it becomes clear that Wittgenstein had tried to obtain the total number from the schematic drawing alone. He has hoped to use for this purpose the expression: 1, ((1)+1), (((1)+1)+ 1), etc., minus the parentheses, 1, 1+1, 1+1+1, etc.! If that had worked out,

it should have been an amorphous sequence where the plus sign "+" would have been used simply as a comma, or internal link, connecting each "one", each summand of the total number. But Wittgenstein concluded in the passage above that this expression doesn't mean anything either. He explains his conclusion by saying that the plus sign would only make a difference if it indicated that the next number is added to a previous total number. Without these additional premises, the plus sign would be just a comma used to separate the groups of "ones", just like the space between the bars from the other example.

Wittgenstein's use of the words "amorphous" and "comma" in the last passage could be an indication that when the plus sign is used as a comma it would favor the display of a purely mechanical "aggregative operation" of putting signs side by side.³⁶

The last "purely aggregative idea" expressed in his second figure, $1 + 1 + 1 + 1$, was not what he wanted either. We could perhaps safely conclude that without the previous recursive steps, explained by an external rule, the internal procedure would become a pure aggregation of signs that could be done in any order. Wittgenstein's conclusion seems to be that the "incremental operation" involves recursion, i.e., it is a general procedure that involves defining a general rule of inference, that from group "n" of signs one can move on to "n+1" (1953, xvi). As Frege had urged, those recursive steps were essential, otherwise one would never go beyond distinct units amorphously arranged in a group that could be rearranged in any pattern or order.

Now points taken together as a group may perhaps fall into some pattern or other like a constellation or may equally arrange themselves somehow or other on a straight line; [...] Patterns produced in this way can be completely different while the number of their elements remains the same. (1953, 53, §41)

Could the pattern $1 + 1 + 1 + 1$ be an *amorphous* sign as Wittgenstein thought? The main problem, as we have anticipated before, was the difference between those three conceptions: "addition as a third element that operates upon the numbers and yields a result", "addition as an operation of joining units with the help of a variable" and "simply grasping an aspect of a figure which displays the actual configuration of bars".

Wittgenstein seemed to have decided at that time to accept the idea that addition demanded external instructions describing how one should proceed, but not the idea of an operation yielding the result. So, in the case of arriving at any ordinal number, the procedure should involve defining the general rule of inference: "n" to "n+1". The difficulty for him is that the simple act of copying a sign many times, separating each instance by a comma (or by any other means) without determining the right sequence of steps did not allow the construction of the series. One would simply remain with that arrangement of signs which could be joined in multiple ways or even lined in any kind of order, as Frege had suggested.

FINAL REMARKS

It is now time to review what we've covered. The first two sections were decisive to our primary goal of setting Wittgenstein's difficulties with the color incompatibility problem into perspective. As we have shown in this first part of the paper, Wittgenstein's first effort to deal with the final phase of GA and to reduce colors to numbers in the *Tractatus* still involved maintaining Frege's ideas of "propositional sense" as "truth-conditions".

From the middle to the end of this paper we had aimed to clarify the exact point where

³⁶ Cf. the passage (1953, 55, §43) quoted from Frege's *Grundlagen* in page 33.

Wittgenstein's ideas went astray, the steps Wittgenstein has taken to supersede them, and his final attempt to solve the problem by returning back to Arithmetic.

Through the analysis of the *Manuscripts* 105 and 106, we confirmed our idea that Wittgenstein's proposal of a new logical form for EPs was an essential part of the development of his solution to the color incompatibility problem. First, the general guideline to analyze away all general terms from EPs was directly connected to this other idea of construing "numbers" as "schemes", developed in detail in the *Manuscripts*. The link between GA's goal of reducing all propositions to a logical product of EP considered as a pure list of names and Wittgenstein's construal of "numbers" as "schemes" is simple and direct. Each scheme figuratively represents what all equinumerous conceptual ranges have in common, but they do so independently from their original concepts (general terms). The other relevant idea for understanding Wittgenstein's construal of "numbers" as "schemes" is that the EPs' constituent elements are held together by an internal relation. This last idea represents a continuation of his discussions in the *Tractatus*. Wittgenstein's construal of *internal/external* relations which hold between the elements of an EP aroused from an old dissatisfaction he had with Frege and Russell's idea of a unique universal conceptual language that would take care of all linguistic practices. The inclusion of formal concepts at the same level as the ordinary ones had always caused a philosophical mistrust in Wittgenstein.

The construal of "numbers" as "schemes" represented a decisive move toward the idea of an "aggregative thinking", though, the very same ideas which had also been criticized by Frege in his *Grundlagen*. Despite his efforts, Wittgenstein eventually concluded that Frege's interpretation of "propositional sense" as "truth conditions" was simply not the correct way to deal with "numerical statements" at all. Detached from their propositional context, those abstract structures should be just a means towards an end: the manipulation of signs and the implementation of the *incremental operation* which generated the series of natural numbers.

The new logical form of EPs proposed by Wittgenstein in the *Tractatus*, but discussed in more detail in the *Manuscripts*, had another critical role to perform. It should also explain how we could have localities and colors determined through the construal of "simple tractarian names" as "numerical coordinates". As we have shown, the two ideas – that of a logical product and that of obtaining the sum of units – are *incompatible* with each other.

We believe one of the important roles of this paper was to review all those connections and obstacles. We believe that our discussion of Wittgenstein's reasons for finally concluding that numbers cannot be generated through a truth-functional procedure might shed some light on our understanding of the color incompatibility problem. A further idea that must be left to be investigated in another paper is Wittgenstein's use of the bar notation in his construal of "numbers" as "schemes". As we've seen, his bar signs would have the task of denoting the units and, at the same time, exemplifying them. The display of sequences of bars should also provide support for the idea of considering numbers as the result of an *incremental operation* upon units. All those results must be further investigated for they have multiple connections and developments within Wittgenstein's later philosophy.

We believe that the most important proposal of this paper though was the idea of focusing on the influence of Frege's upon Wittgenstein's ideas. As we have tried to show above, Frege's construal of "number" as a "predication about concepts", although very illuminating by itself, and even "*a big step that has to be taken*", as Wittgenstein himself describes it, was born in error.³⁷ Frege's oversight was to consider that the logic of propositions and the logical form of

37 Cf. (1976, 166, XVII); (1976, XVII p. 168), (1976, XXVII-I p. 264-65).

subject and predicate were necessary preconditions for understanding mathematical propositions. As Wittgenstein realized at the time in which he faces the problem with color predicates, there was no room for arithmetic taken as an operation applied to elementary propositions, i.e., meaningful propositions. Addition, multiplication, and the very notion of "natural number" as "series construed by the incremental operation" were *not truth-functional operations!* For Wittgenstein in the *Manuscripts*, they were connected by internal and external properties of a distinguished *bar notation* and must be taken out of the propositional context of empirical propositions. Contrary to what Frege thought, Wittgenstein believed that those numerical operations were connected to purely definitional identities, they were "rules of equations", and had the single purpose of exhibiting to us what substitutions could or could not be done.

Since the *Tractatus* Wittgenstein was profoundly convinced that there was a fundamental cleavage between empirical propositions and "numerical statements of mathematics. For him, those latter "statements" must be banished from the meaningful part of the language, which should be composed only by the former. Thus, moving against Frege's logicism, Wittgenstein concluded that the imposition of a logical grammar involving the subject/predicate distinction for construing numerical expressions would bring unnecessary complications. At the time of these *Manuscripts*, his proposal of dropping out the conceptual part and dealing only with concept ranges meant a return to the problem of construing arithmetic, a problem he thought he had solved already.

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Abstract

This paper deals with the so-called "color incompatibility problem", faced by Wittgenstein in the *Tractatus Logico-Philosophicus*, in his unpublished Manuscripts 105 and 106, and in his only published paper – "Some Remarks on Logical Forms" – (SRLF, for now on). Our task will be a twofold one. First, we aspire to show how and why Wittgenstein got stuck into that dilemma. Our second task will be much more specific. We will try to elucidate some details about the impossibility of reducing color predicates to more fundamental units of brightness, chroma, and intensity. We aim to show how and why numbers had to be introduced into the inner structure of elementary propositions. We also want to investigate how he tried to do that through the idea of a "scheme" and a non-truth functional operation. We aim also to present some advantages Wittgenstein gained from using a bar notation for his construal of the "schemes".

Key-words: Wittgenstein, color incompatibility problem, numbers as schemes, recursive operations, early middle period.

Resumo

Este artigo trata do chamado "problema de incompatibilidade de cores", enfrentado por Wittgenstein no *Tractatus Logico-Philosophicus*, em seus Manuscritos inéditos 105 e 106, e em seu único artigo publicado – "Some Remarks on Logical Forms" (SRLF). Nossa tarefa será dupla. Primeiro, pretendemos mostrar como e por que Wittgenstein ficou preso nesse dilema. Nossa segunda tarefa será muito mais específica. Tentaremos elucidar alguns detalhes sobre a impossibilidade de reduzir predicados de cor a unidades mais fundamentais de brilho, croma e intensidade. Nosso objetivo é mostrar como e por que os números tiveram que ser introduzidos na estrutura interna das proposições elementares. Também queremos investigar como Wittgenstein tentou fazer isso através da ideia de um "esquema" e uma operação não ver-funcional. Pretendemos também apresentar algumas vantagens que Wittgenstein obteve ao usar uma notação de barra para sua construção dos "esquemas".

Palavras-chave: Wittgenstein, problema de incompatibilidade de cores, números como esquemas, operações recursivas, período intermediário inicial.